LINEAR MAPS

1 Back to previous definitions

Definition.

Consider two sets E and F and a mapping $f : E \longrightarrow F$ (a linear map or any other mapping). We say that:

1. *f is injective if:*

$$\forall\,(u,u') \in E^2,\ f(u) = f(u') \implies u = u'$$

2. *f is surjective if:*

$$\forall\, v \in F,\ \exists\, w \in E\ v = f(w)$$

Definition.

Let E and F be two vector spaces over $\mathbb{R}$ and $f \in \mathcal{L}(E,F)$.

1. *The kernel of f is the set:*

$$Ker(f) = \{u \in E,\ f(u) = 0_F\}$$

2. *The image of f is the set:*

$$Im(f) = \{v \in F,\ \exists\, w \in E,\ v = f(w)\}$$

Remarks:

1. Let $u \in E$. Writing that $u \in \mathrm{Ker}(f)$ means that $f(u) = 0_F$:

$$\forall u \in E,\ u \in \mathrm{Ker}(f) \iff f(u) = 0_F$$

2. Let $v \in F$. Writing that $v \in \mathrm{Im}(f)$ means that we can find a relation $v = f(w)$ with $w \in E$.

$$\forall v \in F,\ v \in \mathrm{Im}(f) \iff v = f(w) \text{ with } w \in E \iff \exists\, w \in E,\ v = f(w)$$

3. $\mathrm{Ker}(f) = f^{-1}(\{0_F\})$ and $\mathrm{Im}(f) = \{\, f(u),\ u \in E\,\} = f(E)$.

2 The kernel and the image are vector spaces

Proposition (Kernel and image).

Let E and F be two vector spaces over $\mathbb{R}$ and $f \in \mathcal{L}(E,F)$.

1. *$Ker(f)$ is a linear subspace of E.*
2. *$Im(f)$ is a linear subspace of F.*

Proof:

1. • By definition, $\text{Ker}(f) \subset E$. Since f is a linear map from E to F, we know that $f(0_E) = 0_F$, that is, $0_E \in \text{Ker}(f)$.

   • Let $(u, v) \in \big(\text{Ker}(f)\big)^2$ and $\alpha \in \mathbb{R}$.

   $$\begin{aligned} f(\alpha\, u + v) &= \alpha\, f(u) + f(v) \text{ because } f \text{ is a linear map} \\ &= \alpha\, 0_F + 0_F \\ &= 0_F \end{aligned}$$

   Thus, $\alpha\, u + v \in \text{Ker}(f)$. $\text{Ker}(f)$ is hence a linear subspace of E.
2. • By definition, $\text{Im}(f) \subset F$. Since f is a linear map from E to F, $0_F = f(0_E)$. Thus, $0_F \in \text{Im}(f)$.

   • Let $(v, v') \in \big(\text{Im}(f)\big)^2$ and $\alpha \in \mathbb{R}$. Then:

   $$v \in \text{Im}(f) \iff \exists w \in E,\ v = f(w) \qquad \text{and} \qquad v' \in \text{Im}(f) \iff \exists w' \in E,\ v' = f(w')$$

   We hence get:

   $$\begin{aligned} \alpha\, v + v' &= \alpha\, f(w) + f(w') \\ &= f(\alpha\, w + w') \text{ since } f \text{ is a linear map} \end{aligned}$$

   This proves that $\alpha\, v + v' \in \text{Im}(f)$. $\text{Im}(f)$ is hence a linear subspace of F.

3 Connection with the injectivity and surjectivity

Proposition (Characterizing injective and surjective linear maps).

Let E and F be two vector spaces over $\mathbb{R}$ and $f \in \mathcal{L}(E, F)$. Then:

1. *f injective $\iff$ $Ker(f) = \{0_E\}$.*
2. *f surjective $\iff$ $Im(f) = F$.*

Proof:

1. $\boxed{\Longrightarrow}$ Assume that f is injective. We hence know that

   $$\forall\, (u, u') \in E^2,\ f(u) = f(u') \implies u = u'$$

   • Let $u \in \text{Ker}(f)$.
   $f(u) = 0_F$ and $0_F = f(0_E)$ (because f is a linear map) $\implies f(u) = f(0_E) \implies u = 0_E$ using the injectivity definition. This proves that $\text{Ker}(f) \subset \{0_E\}$.

   • Since $\{0_E\} \subset \text{Ker}(f)$, we get: $\text{Ker}(f) = \{0_E\}$.

   $\boxed{\Longleftarrow}$ Assume that $\text{Ker}(f) = \{0_E\}$.

   Let $(u, u') \in E^2$ such that $f(u) = f(u')$. Then:

   $$\begin{aligned} f(u) = f(u') &\implies f(u) - f(u') = 0_F \\ &\implies f(u - u') = 0_F \text{ because } f \text{ is a linear map} \\ &\implies u - u' \in \text{Ker}(f) \\ &\implies u - u' \in \{0_E\} \text{ because } \text{Ker}(f) = \{0_E\} \\ &\implies u - u' = 0_E \\ &\implies u = u' \end{aligned}$$

   f is hence injective.
2. For the surjectivity:

   $$\begin{aligned} f \text{ surjective} &\iff \forall\, v \in F,\ \exists\, u \in E\ v = f(u) \\ &\iff \forall\, v \in F,\ v \in \text{Im}(f) \\ &\iff F \subset \text{Im}(f) \\ &\iff F = \text{Im}(f) \text{ since the inclusion } \text{Im}(f) \subset F \text{ is always true} \end{aligned}$$